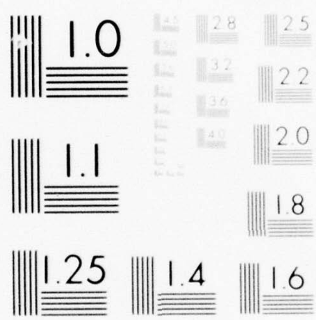
AD-A044 410 TEXAS UNIV AT AUSTIN CENTER FOR CYBERNETIC STUDIES F/G 12/1
LEGACY OF THE PAPER BY KURT FRIEDRICHS ENTITLED EIN VERFAHREN DER
AUG 77 W T REID N00014-75-C-0569
UNCLASSIFIED CCS-300 NL

1 OF 1
AD A044410

END
DATE
FILMED
10-77
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963-A

ADA044410

(12)

CENTER FOR CYBERNETIC STUDIES

The University of Texas
Austin, Texas 78712

6 9 14

Research Report CCS-300

LEGACY OF THE PAPER BY KURT FRIEDRICHS ENTITLED EIN VERFAHREN DER VARIATIONSRECHNUNG, DAS MINIMUM EINES INTEGRAL ALS DAS MAXIMUM EINES ANDEREN AUSDRUCKES DARZUSTELLEN†

by

10

William T. Reid

11 August 1977 12 34p.

†Göttingen Nachrichten, Math. -Phys. (1929), 13-20.

15

This research was partly supported by Project NR047-021, ONR Contract N00014-75-C-0569 with the Center for Cybernetic Studies, The University of Texas. Reproduction in whole or in part is permitted for any purpose of the United States Government.

CENTER FOR CYBERNETIC STUDIES

A. Charnes, Director
Business-Economics Building, 203E
The University of Texas
Austin, Texas 78712
(512) 471-1821

1473
406 197

1B

Abstract

For certain aspects of the calculus of variations and optimal control theory, the paper of Kurt Friedrichs listed in the title of this report has had a strong influence, both directly and indirectly. The present paper is devoted to a brief historical survey of this influence, especially in the area of convex analysis. In addition, a translation of Friedrichs' paper is appended to this report.

1. Introduction. Recently Professor Charnes called my attention to the paper of Friedrichs referenced [15], of which I was not aware, although I was cognizant of various more recent papers whose lineage may be traced back directly or indirectly to this paper. Following are some comments on the influence of this paper on the calculus of variations and control theory. No claim is made as to the completeness of the presented list of references. Moreover, the author does not pretend to be conversant with the details of all the papers listed in this survey. Also appended to this report is a translation of Friedrichs' paper.

In the variational study of extrema wide use is made of the fact that there is no qualitative distinction between minima and maxima. *Indeed, if* $J_o: X \to R$ is a real-valued function on an abstract space X which has a minimum m on X at a value x_o, (i.e., $m = J_o(x_o) \leq J_o(x)$ for all $x \in X$), then $J = -J_o$ is a real-valued function on X which has a maximum $-m$ on X at x_o, (i.e., $-m = J(x_o) \geq J(x)$ for all $x \in X$). In abstract context, the basic contribution of Friedrichs [15] was to associate with a particular integral variational functional $J_o: X \to R$ a related, (variously called "reciprocal", "dual", "conjugate") variational functional $J_1: X_1 \to R$, where X_1 is a second space, in general different from X, such that if J_o possesses a minimum m on X at a value x_o then J_1 possesses at a value $x_1 \in X_1$ a maximum value on X_1 which is also equal to m. In this situation, consideration of extremizing sequences for J_o and J_1 yields both upper and lower bounds for the

extremum value m. In the introduction to [15] Friedrichs states that his attention to the problem was directed by the paper [39] of Trefftz, wherein there is associated with the Ritz method a companion method to obtain both upper and lower bounds, with particular application to a problem in torsional rigidity.

2. Relatively early influences of Friedrichs' paper. Soon after the appearance of Friedrichs' paper [15] it was applied to certain problems in elasticity theory by Basu [5, 6]. The results of [15] were also expounded in the second edition of Courant-Hilbert [8i; §9 of Ch. IV, pp. 199-209], which appeared in 1931; indeed, this discussion discriminates between the non-singularity aspects of Friedrichs' transformation that pertain to the stationary character of the involved functionals, and other (strong convexity) aspects that provide the extremizing properties of the functionals.

In his 1942 address [9; Sec. II. 4] on variational procedures for the solution of problems of equilibrium and vibration, Courant again emphasized the mathematical concept of representing the minimum value of a given variational problem as a maximum value of an associated variational problem. References to Friedrichs' method also appeared in books on elasticity theory, notably in the first edition of Sokolnikoff [36; Sec. 71]. Also in the years 1947-49 there appeared papers [37], [10], [11], [12], [18] presenting work of Prager, Synge, Diaz, Weinstein and Greenberg on variational problems related to that of Friedrichs [15], but in general employing alternate methods of treatment.

In the 1953 English edition of Courant-Hilbert the method of Friedrichs is again elaborated upon [8ii; §9 of Ch. IV, pp. 231-242]. Moreover, there is added a new section on reciprocal quadratic variational problems [8ii; §11 of Ch. IV, pp. 252-257], presenting an abstract analysis of such quadratic problems. In particular, attention is directed to work of Synge

on a geometric interpretation from which one may deduce an estimate of the distance of approximations from exact solutions. For a discussion of Synge's method of the hypercircle see [38], and references presented therein. In the general area of bilateral estimates of extremum values one should also include the "intermediate problem" work of Weinstein, dating from the late 1930's. In this procedure the determination of upper bounds of eigenvalues, which in many cases may be computed by the Rayleigh-Ritz method, is supplemented by the consideration of intermediate problems with weakened prescribed conditions, leading to lower bounds for the eigenvalues, (see Gould [17], and Weinstein and Stenger [40]).

In his study of extremal problems for bounded analytic functions in a multiply connected domain, Lax [23] utilized certain conjugate variational problems which in the abstract resulted from the fact that by an application of the Hahn-Banach theorem it follows that if S is a linear manifold in a normed linear space X and $x \in X$, then the minimum of $\{|y-x| : y \in S\}$ is equal to the maximum of $|\ell(x)|$ for all continuous linear functionals ℓ of norm one which vanish on S. At the end of the introduction to [23] Lax points out that this abstract principle includes the results of the above cited §11 added in the English edition of Courant-Hilbert, and proceeds with the comment: "On the other hand, the abstract principle itself can be regarded as a special case of pairs of conjugate extremal problems described by Friedrichs. This is especially clear from an abstract version of the Friedrichs principle that I have recently found, and which I hope to describe at another occasion."

Recently the author has inquired of Lax as to his abstract version of the Friedrichs principle, but to date has received no reply.

In the 1962 book of Funk [16] there is a rather lengthy discussion, (Ch. VIII, pp. 498-531), of the principle of Friedrichs and its applications in elasticity theory.

3. The central role of convexity. In order to facilitate later discussion, at this stage we formulate in classical terms a variational problem, which for $n=1$ reduces to the first problem considered by Friedrichs [15]. Consider the fixed end-point problem of minimizing the integral functional

$$J_o[y] = \int_{x_o}^{x_1} f(x, y(x), y'(x))dx \tag{3.1}$$

in the class of n-dimensional vector functions $y(x) = y_\alpha(x))$, $(\alpha = 1, \ldots, n)$, which are continuous and piecewise continuously differentiable on $[x_o, x_1]$, with elements $(x, y(x), y'(x))$, $x \epsilon [x_o, x_1]$, in a given region R in $(x, y, r) = (x, y_1, \ldots, y_n, r_1, \ldots, r_n)$ space, and satisfying the end-conditions

$$y(x_o) = y^o, \quad y(x_1) = y^1 . \tag{3.2}$$

It is assumed that for $x_o \leq x \leq x_1$ each of the sections $R_x = \{(y, r):(x, y, r) \epsilon R\}$ is a convex subset of (y, r)-space. Moreover, it is supposed that on R the real-valued integrand function $f(x, y, r) = f(x, y_1, \ldots, y_n, r_1, \ldots, r_n)$ is continuous and has continuous partial derivatives of the first two orders with respect to the y_α, r_α, $(\alpha = 1, \ldots, n)$, while for $(x, y, r) \epsilon R$ the $2n \times 2n$ real symmetric matrix

$$\begin{bmatrix} f_{rr} & f_{ry} \\ f_{yr} & f_{yy} \end{bmatrix} = \begin{bmatrix} f_{r_\alpha r_\beta} & f_{r_\alpha y_\beta} \\ f_{y_\alpha r_\beta} & f_{y_\alpha y_\beta} \end{bmatrix}, \quad (\alpha, \beta = 1, \ldots, n) \tag{3.3}$$

is positive definite.

Under these hypotheses the vector equations

$$\begin{matrix} f_r(x, y, r) = z, \\ f_y(x, y, r) = w, \end{matrix} \qquad \left\{ \begin{matrix} f_{r_\alpha}(x, y, r) = z_\alpha, \\ f_{y_\alpha}(x, y, r) = w_\alpha, \end{matrix} \quad (\alpha = 1, \ldots, n) \right\} \tag{3.4}$$

define a 1-1 continuously differentiable mapping of R onto a region Δ of (x, z, w)-space. If

$$(3.5)\qquad \begin{aligned} r &= R(x, z, w) \\ y &= Y(x, z, w), \end{aligned} \qquad \left\{ \begin{aligned} r_\alpha &= R_\alpha(x, z, w), \\ y_\alpha &= Y_\alpha(x, z, w), \end{aligned} \quad (\alpha = 1, \ldots, n) \right\}$$

denote the solution of (3.5) for $(x, z, w) \in \Delta$, and $\Phi(x, z, w)$ is defined by the Legendre transform

$$(3.6)\qquad \Phi(x, z, w) = z^* r + w^* y - f(x, y, r)\Big|_{y=Y(x,z,w),\ r=R(x,z,w)},$$

(where in general v^* is used to denote the transpose of a vector V), then the reciprocal functional $J_1[z]$ of Friedrichs [15] is the negative of the functional

$$(3.7)\qquad J_2[z] = \int_{x_o}^{x_1} \Phi(x, z(x), z'(x))dx - [z^*(x_1)y^1 - z^*(x_o)y^o].$$

Consequently, the statement that on the class of continuous and piecewise continuously differentiable $z(x)$, $x \in [x_o, x_1]$, with elements $(x, z(x), z'(x)) \in \Delta$ the maximum of $J_1[z]$ is equal to the minimum of $J_o[y]$ on the above described class of $y(x)$ satisfying (3.2) is equivalent to the statement that on the respective classes the *minimum* of $J_2[z]$ is equal to the negative of the minimum of $J_o[y]$.

In the particular case wherein f involves y only linearly, so that we may write

$$(3.8)\qquad f(x, y, r) = g(x, r) + k^*(x)y,$$

the positive definiteness of (3.3) is replaced by the positive definiteness of the n x n real symmetric matrix

(3.9) $\quad f_{rr}(x, y, r) = g_{rr}(x, r) = [g_{r_\alpha r_\beta}(x, r)]$, $(\alpha, \beta = 1, \ldots, n)$.

Correspondingly, (3.4) is replaced by

(3.10) $\quad g_r(x, r) = z$

with solution $r = R(x, z)$, and

(3.11) $\quad \Phi(x, z) = z^* r - g(x, r) \Big|^{r=R(x,z)}$.

Finally, in Section III of [15] Friedrichs discusses a quadratic double integral problem, for which reference is made to the appended translation.

The point of principal interest is that under the above described differentiability conditions the positive definiteness of the 2n x 2n matrix (3.3) throughout R is equivalent to the condition that f is strongly convex in (y, r) on each section R_x. *Correspondingly, the positive definiteness of* (3.9) is equivalent to the strong convexity of f in r on each $R_{x,y} = \{r : (x, y, r) \in R\}$.

Now conditions involving the positive definiteness of the matrix $f_{rr}(x, y, r)$ are of long-standing in the calculus of variations. For a variational problem involving (3.1) the satisfaction of this condition along a given extremal arc is known as the strengthened, (or sufficient condition form of the) Legendre condition, and appears in standard sufficiency theorems of the classical calculus of variations. The assumption that the matrix $f_{rr}(x, y, r)$ be positive definite throughout the entire region R of admissible elements has frequently been called a condition of regularity. In particular, this condition implies a definiteness of the Weierstrass E-function, and the lower semi-continuity of the integral functional $J_0[y]$ in certain senses. For a discussion of the significance of the convexity of f in r, and indeed the

recasting of the entire Hamiltonian concepts in terms of convexity, the reader is referred to Young [41; Chs. IV and V in particular]. In this connection it is to be remarked that only quite recently has Cesari [7] proved that if f is continuous in (x, y, r) for (x, y) in a given closed subset A of (x, y)-space and r arbitrary, while $\mathcal{T}$ denotes the class of absolutely continuous vector functions $y(x) = (y_\alpha(x))$, $x \varepsilon [x_0, x_1]$, with graph in A and $f(x, y(x), y'(x))$ Lebesgue integrable, (finite or infinite), on $[x_0, x_1]$, and (m) denotes the mode of convergence of sequences $\{y^{(k)}(t)\}$ of elements in $\mathcal{T}$ in the sense of weak convergence of $\{y^{(k)'}(t)\}$ in the Lebesgue space $L[x_0, x_1]$ and uniform convergence of $\{y^{(k)}(t)\}$ on $[x_0, x_1]$, then a necessary and sufficient condition for the lower semi-continuity of $J_0[y]$ with respect to mode (m) at each $y \varepsilon \mathcal{T}$ is that for each $(x, y) \epsilon A$ the function f is convex in r.

Modern appreciation of the central role of convexity in the study of extremum problems received initial impetus from the concept of conjugate convex functions introduced by Fenchel [14], and the fact that Fenchel's conjugate correspondence for convex functions may be viewed as a generalization of the classical Legendre transformation. For a detailed discussion of this concept and the fact that it enables one to treat variational problems devoid of differentiability assumptions, see Rockafellar [31; Secs. 12 and 16, in particular]. Attention will be limited here to listing papers dealing with "dual" or "reciprocal" problems under conditions corresponding somewhat to the strong convexity assumption of Friedrichs expressed by the positive definiteness of the matrix (3.3) in the problem specifically formulated above.

In this connection there are the papers of Hanson [21], Pearson [26, 27], Kreindler [22], Mond and Hanson [25]. However, the work of Rockafellar as presented in [32, 33, 34, 35] appears to be more intimately related to the work of Friedrichs. In particular, in these papers Rockafellar is concerned with generalized variational problems of Bolza form, wherein various restraints are incorporated through the use of $+\infty$ as a possible functional value for the involved "integrand functions" and "boundary functionals". Hager and Mitter [20] have recently considered dual problems that are somewhat different from those treated by Rockafellar [34], and wherein the constraints are given explicitly by inequalities. Also, for control problems of Bolza in Hilbert spaces Barbu [2, 3] has used methods of convex analysis and of maximal monotone operators to obtain results corresponding to some of those of Rockafellar in the papers cited above.

For a class of Bolza type problems involving functions which satisfy differentiability conditions of a more classical nature and a type of "semi-local convexity", sufficient conditions for a global extremum have been given recently by Ewing [13]. A somewhat related sufficiency proof of an absolute minimum for a non-parametric variational problem has been presented by Reid [30].

4. <u>Other remarks</u>. This section is devoted to comments on two types of problems concerned with the Euler equations for variational functionals that are reciprocal in the sense of Friedrichs [15].

Firstly, in the study of oscillation phenomena for ordinary differential equations there has appeared the concept of reciprocal equations. Specifically for $p(x)$ and $q(x)$ continuous real-valued functions which are non-zero on a given interval I on the real line, with the differential equation

$$(4.1) \qquad [p(x)y'(x)]' - q(x)y(x) = 0$$

there has been associated the "reciprocal equation"

$$(4.2) \qquad \left[\frac{1}{q(x)} z'(x)\right]' - \frac{1}{p(x)} z(x) = 0 ,$$

(see Potter [28], and Barrett [4]. If

$$(4.3) \qquad f(x, y, r) = \frac{1}{2}[p(x)r^2 + q(x)y^2] ,$$

then (4.1) is the Euler equation for the variational integral (3.1), and (4.2) is the Euler equation for the associated integral of (3.7), wherein

$$\Phi(x, z, w) = \frac{1}{2}\left[\frac{1}{q(x)} w^2 + \frac{1}{p(x)} z^2\right] .$$

For integral functionals (3.1) with integrand a quadratic form

$$(4.4) \qquad f(x, y, r) = r^*[R(x)r + Q(x)y] + y^*[Q^*(x)r + P(x)y]$$

in $(y, r) = (y_1, \ldots, y_n, r_1, \ldots, r_n)$ and $R(x)$ nonsingular, with the canonical form of the Euler equation in vector functions $(u;v) = (u_1, \ldots, u_n; v_1, \ldots, v_n)$. Reid [29] associated a related problem, that he called the "obverse problem", and which had the property that $(u;v)$ was a solution of the canonical system for the original problem if and only if $(v;u)$ was a solution of the obverse

problem. If for the variational function (3.1) with integrand (4.4) we have in addition to the non-singularity of $R(x)$ that the $2n \times 2n$ matrix function

$$\begin{bmatrix} R(x) & Q(x) \\ Q^*(x) & P(x) \end{bmatrix}$$

is non-singular for all x then the obverse system of [29] is indeed the canonical form of the Euler equations for the corresponding reciprocal problem of Friedrichs. An alternate equivalent form of associated problem, wherein $(u;v)$ is a solution of the primal differential system if and only if $(-v;u)$ is a solution of the associated system, has been introduced by Ahlbrandt [1].

Finally, it is to be noted that for a double integral non-parametric variational integral

(4.5) $$J[z] = \iint_G F(z_x, z_y)\,dxdy,$$

Haar [19] introduced the concept of an associated "adjoint" problem, wherein an extremal surface for (4.5) determines a corresponding "adjoint extremal surface" of the adjoint problem. In particular, Haar's basic assumptions on the integrand function $F(p,q)$ involved the non-zero character of $F_{pp}F_{qq} - F_{pq}^2$ and $F - pF_p - qF_q$ for all admissible p, q. Subsequently, Mickle [24] showed that under suitable non-singularity conditions the adjoint problem of Haar was but one of a set of twenty-four problems associated with (4.5) through like transformations, and that the twenty-four transformations possessed certain group properties. As variational problems of this

sort are in nature somewhat like the double integral problem considered in the last section of [15], there arises the question as to whether or not there exist pertinent relationships between the respective reciprocal problems of Friedrichs for a given problem (4.5) and an associated problem of Mickle. As far as the author is aware, such questions have not been considered.

REFERENCES

1. Ahlbrandt, C.D., Equivalent Boundary Value Problems for Self-adjoint Differential Systems, J. Differential Eqs., 9(1971), 420-435.

2. Barbu, V., Convex Control Problems of Bolza in Hilbert Spaces, SIAM J. Control, 13(1975), 754-771.

3. Barbu, V., On the Control Problem of Bolza in Hilbert Spaces, SIAM J. Control, 13, (1975), 1062-1076.

4. Barrett, J.H., Disconjugacy of Second Order Linear Differential Equations with Non-negative Coefficients, Proc. Amer. Math. Soc., 10(1959), 552-561.

5. Basu, N.M., On an Application of the New Methods of the Calculus of Variations to Some Problems in the Theory of Elasticity, Phil. Mag. (7), 10(1930), 886-896.

6. Basu, N.M., On the Torsion Problem of the Theory of Elasticity, Phil. Mag. (7), 10(1930), 896-904.

7. Cesari, L., A Necessary and Sufficient Condition for Lower Semi-Continuity, Bull. Amer. Math. Soc., 80(1974), 467-472.

8. Courant, R., and Hilbert, D., (i) Methoden der Mathematischen Physik I, 2nd ed., Springer, Berlin 1931; (ii) Methods of Mathematical Physics, (translated from German original and revised), Interscience, N.Y., 1953.

9. Courant, R., Variational Methods for the Solution of Problems of Equilibrium and Vibration, Bull. Amer. Math. Soc., 49(1943), 1-23.

10. Diaz, J.B., and Weinstein, A., Schwarz' Inequality and the Methods of Rayleigh-Ritz and Trefftz, J. Math. Physics, 26(1947), 133-136.

11. Diaz, J.B., and Weinstein, A., The Torsional Rigidity and Variational Methods, Amer. J. Math., 70(1948), 107-116.

12. Diaz, J.B., and Greenberg, H.J., Upper and Lower Bounds for the Solutions of the First Boundary Problem of Elasticity, Quart. Appl. Math., 6(1948), 326-331.

13. Ewing, G., Sufficient Conditions for Global Minima of Suitably Convex Functionals for Variational and Control Theory, SIAM Review, 19(1977), 202-220.

14. Fenchel, W., On Conjugate Convex Functions, Canadian J. Math., 1(1949), 73-77.

15. Friedrichs, K., Ein Verfahren der Variationsrechnung, das Minimum eines Integral als das Maximum eines änderen Ausdruckes darzustellen, Göttingen Nachrichten, Math.-Phy. Klasse, (1929), 13-20.

16. Funk. P., Variationsrechnung und ihre Anwendung in Physik und Technik, Grundlehren Math. Wiss., Bd. 94, Springer-Verlag, Berlin, 1962.

17. Gould, S. H., Variational methods for eigenvalue Problems, 1st ed., 1955; 2nd rev. ed. 1966, Univ. Toronto Press, Toronto, Canada.

18. Greenberg, H. J., The Determination of Upper and Lower Bounds for the Solution of the Dirichlet Problem, J. Math. Physics, 27(1948), 161-182.

19. Haar, A., Über Adjungierte Variationsprobleme und Adjungierte Extremalflachen, Math. Ann., 100(1928), 481-502.

20. Hager, W.W., and Mitter, S.K., Lagrange Duality for Convex Control Problems, SIAM J. Control and Optimization, 14(1976), 843-856.

21. Hanson, M.A., Bounds for Functionally Convex Optimal Control Problems, J. Math. Anal. Appl. 8(1964), 84-89.

22. Kreindler, E., Reciprocal Optimal Control Problems, J. Math. Anal. Appl., 14(1966), 141-152.

23. Lax, P.D., Reciprocal Extremal Problems in Function Theory, Comm. Pure Appl. Math., 8(1955), 437-453.

24. Mickle, E.J., Associated Double Integral Variational Problems, Duke Math. J., 9(1942), 208-227.

25. Mond, B., and Hanson, M.A., Duality for Variational Problems, J. Math. Anal. Appl., 18(1967), 355-364.

26. Pearson, J.D., Reciprocity and Duality in Control Programming Problems, J. Math. Anal. Appl., 10(1965), 388-408.

27. Pearson, J.D., Duality and a Decomposition Technique, SIAM J. Control, 4(1966), 164-172.

28. Potter, R.L., On Self-adjoint Differential Equations of the Second Order, Pacific J. Math., 3(1953), 467-491.

29. Reid, W.T., Monotoneity Properties of Solutions of Hermitian Riccati Matrix Differential Equations, SIAM J. Math. Anal., 1(1970), 195-213.

30. Reid, W.T., An Elementary Sufficiency Proof of an Absolute Minimum for a Non-parametric Variational Problem, J. Opt. Theory Appl., 18(1976), 333-347.

31. Rockafellar, R.T., Convex Analysis, Princeton Univ. Press, Princeton, N.J., 1970.

32. Rockafellar, R.T., Conjugate Convex Functions in Optimal Control and the Calculus of Variations, J. Math. Anal. Appl., 32 (1970), 174-222.

33. Rockafellar, R.T., Existence and Duality Theorems for Convex Problems of Bolza, Trans. Amer. Math. Soc., 159(1971), 1-40.

34. Rockafellar, R.T., State Constraints in Convex Control Problems of Bolza, SIAM J. Control, 10(1972), 691-715.

35. Rockafellar, R.T., Dual Problems of Lagrange for Arcs of Bounded Variation, Calculus of Variations and Control Theory, (Proceedings of a conference held at the MRC, Univ. of Wisconsin, Madison, Wisconsin, Sept. 1975; edited by D.L. Russell), Academic Press, 1976.

36. Sokolnikoff, I.S., Mathematical Theory of Elasticity, 1st Ed., Mc-Graw-Hill, N.Y., 1946; 2nd ed., 1956.

37. Synge, J.L., and W. Prager, Approximations in Elasticity Based on the Concept of Function Space, Quart. Appl. Math., 5(1947), 241-269.

38. Synge, J.L., The Hypercircle in Mathematical Physics, Cambridge Univ. Press, London, 1957.

39. Trefftz, E., Ein Gegenstück zum Ritzschen Verfahren, Proc. 2nd International Congress for Applied Mechanics, Zürich (1926), 131-137.

40. Weinstein, A., and Stenger, W., Methods of Intermediate Problems for Eigenvalues: Theory and Ramifications, Academic Press, N.Y., 1972.

41. Young, L.C., Lectures on the Calculus of Variations and Optimal Control Theory, W.B. Saunders Co., Philadelphia, Pa., 1969.

5. Translation of Friedrichs' paper.

A METHOD OF THE CALCULUS OF VARIATIONS, WHEREBY THE MINIMUM OF AN INTEGRAL IS PRESENTED AS THE MAXIMUM OF ANOTHER EXPRESSION

By Kurt Friedrichs, Aachen

Presented by R. Courant in the session of December 7, 1928.

In the numerical solution of variational problems by the Ritz method it is important to estimate the accuracy of approximation[1]. E. Trefftz has given a method -- indeed, for the Dirichlet problem and related topics -- whereby the solution of the calculus of variations problem is approximated in such a manner that the minimum value is approached from below. Consequently, through the combined application of this method and that of Ritz the minimum value is bounded from both sides. In the following the same goal is achieved, and in a manner more general than that of Trefftz. Very generally one may associate with a given minimum problem a maximum problem, whose maximum value is equal to the minimum value of the original problem. The underlying principle is essentially a Legendre transformation. For example, to the variational problem for the solution of the potential equation with prescribed boundary conditions there is associated a problem for the conjugate potential function. Also, with the aid of a Legendre transformation, in elasticity theory one associates with the principle of "virtual displacement" the so-called Castigliano principle of "minimum work of deformation".

In order to proceed with assurance that the new formally constructed variational problem does admit a maximum, there must be imposed certain strong conditions of definiteness, which moreover frequently remain satisfied by modifications of the variational expression.

For numerical consideration the following fact is also of importance: if for the initial problem the boundary values of the function are prescribed, such is not the case for the associated problem; for it, the boundary conditions appear as "natural" ones. The first case has the disadvantage that one is restricted in the choice of approximation functions; in the second case, empirically the convergence is very slow. However, one now has the possibility of choosing which handicap he will accept.

I. Variational problems in one variable.

Firstly, we consider the problem of minimizing the integral

(1) $$\int_{x_0}^{x_1} F(u', u, x)dx,$$

with the functions u(x) satisfying prescribed boundary conditions. Moreover, it is to be remarked that the following discussion can be carried out when boundary conditions of another type are present, but we do not discuss this aspect separately.

The fact that u'(x) is the derivative of u(x)[4)] is formulated as an individual equation

(2) $$\frac{du}{dx} = u' .$$

The boundary conditions are expressed as:

(3) $u(x_0) = \bar{u}_0$, $u(x_1) = \bar{u}_1$, or simply, $u = \bar{u}$ for $x=x_0$ and $x=x_1$,

where $\bar{u}_0$, $\bar{u}_1$ are arbitrary given values.

Finally, let

(4) $F_{u'u} > 0$, [This should read $F_{u'u'} > 0$]

(5) $\Delta = F_{u'u'}F_{uu} - F_{uu'}^2 > 0$.

We suppose that the minimum of (1) is provided by a function u(x) which has continuous derivatives of the first two orders, and has elements (u'(x), u(x), x) interior to B. It satisfies the Euler equation

(6) $$\frac{d}{dx} F_{u'} = F_u .$$

We now introduce new functions p(x), p'(x) through the equations

(7) $F_{u'} = p$, $F_u = p'$,

by which, in view of (5), u and u' are determined as functions of p and p'. With these functions we construct a new integrand

(8) $\qquad \Phi(p', p, x) = pu' + p'u - F(u', u, x),$

for which

(9) $\qquad \Phi_{p'} = u, \quad \Phi_p = u' .$

As a new variational problem we now consider

(10) $\qquad \int_{x_o}^{x_1} \Phi dx - p\bar{u} \Big|_{x_o}^{x_1} = \text{Min.}$ [5)]

on the class of all continuous functions p(x) and p'(x) [6)], not restricted by boundary conditions, and subject only to the subsidiary equation

(11) $\qquad \frac{dp}{dx} = p'.$

We maintain that since u(x) and u'(x) are elements of a solution of the minimum problem involving (1) the corresponding functions p(x), p'(x) of (7) are elements of a minimum for the expression (10). First of all, in view of (6) equation (11) is satisfied. The new Euler equation

(12) $\qquad \frac{d}{dx} \Phi_{p'} = \Phi_p ,$

is, in view of (9), none other than the subsidiary condition (2), and is also satisfied. The natural boundary condition

(13) $\qquad \Phi_{p'} - \bar{u} = 0 \qquad \text{for } x=x_o \text{ and } x=x_1 ,$

is none other than the boundary condition (3). Consequently the expression (10) is stationary. That it is indeed a minimum follows from the positiveness of the second variation of (10) for all (p'(x), p(x), x) in the region B*. In view of (4), (5), from (9), (7) one obtains

$$(14) \qquad \Phi_{p'p'} = \frac{1}{\Delta} F_{u'u'} > 0$$

$$(15) \qquad \Phi_{p'p'}\Phi_{pp} - \Phi^2_{pp'} = \frac{1}{\Delta} > 0$$

The minimum value for the new problem (10) is

$$\int_{x_o}^{x_1} (pu'+p'u-F)dx - pu\Big|_{x_o}^{x_1} = -\int_{x_o}^{x_1} Fdx,$$

which is the negative of the minimum value for problem (1).

II. Appendices

1. If the strong definiteness condition (5) is not satisfied, but the greatest values of $-F_{uu}$ and $|F_{uu'}|$ are sufficiently small in relation to the smallest value of $F_{u'u'}$, the condition (5) can be attained by the addition[7] of an integral

$$\int_{x_o}^{x_1} \frac{d}{dx} G(u, x)dx$$

with G appropriately chosen -- perhaps of the form $G \equiv (\alpha x + \beta)u^2$.

2. Without knowledge of the existence of a solution of (1), it follows that the sum of the greatest lower bounds of (1) and (10) is non-negative. Indeed, for arbitrary independent admissible functions[8] $u(x)$, $p(x)$ with $u' = du/dx$, $p' = dp/dx$,

$$\int_{x_o}^{x_1} [F(u) + \Phi(p)]dx - pu \Big|_{x_o}^{x_1}$$

$$= \int_{x_o}^{x_1} [F(u) - F(v) + vF_u(v) + v'F_{u'}(v) - uF_u(v) - u'F_{u'}(v)]dx,$$

wherein we set $v = \Phi_{p'}$, $v' = \Phi_p$ and $p = F_{u'}(v)$, $p' = F_u(v)$. The integrand is none other than the second order remainder in the expansion of F(u) with respect to u', u in a neighborhood of v', v, which in view of (4), (5) is non-negative and equal to zero if and only if $u = v$, $u' = v'$; that is, if and only if u(x) and p(x) are solutions of the respective problems.

3. We consider also the degenerate case $F_{uu} = F_{uu'} = 0$, wherein we may set

(16) $\qquad F = G(u', x) + k(x)u.$

Here we require only the condition (4); that is, $G_{u'u'} > 0$. The Legendre transformation now becomes simply

(17) $\qquad F_{u'} = p, \ \Phi(p, x) = pu' - G, \ \Phi_p = u'$,

where we may express u' in terms of p and conversely. The function $p(x)$ thus arising from a solution $u(x)$ of (1) renders (10) a minimum in the class of all functions $p(x)$ satisfying the subsidiary condition

(18) $$\frac{dp}{dx} = k .$$

(p' does not appear in the expression (10)). Indeed, (18) permits only variations of the form p+constant, and the first variation of (10) is

$$\int_{x_o}^{x_1} \Phi_p dx - \bar{u} \Big|_{x_o}^{x_1} = 0 ,$$

in view of $\Phi_p = u'$, while the second variation is positive in view of (4).

4. When we distinguish for a variational problem the constraint conditions: u' is the derivative of u, u assumes given boundary values, and the variational conditions : Euler equation, natural boundary (transversality) conditions, the preceding discussion yields the result that the constraint conditions of the original problem are the variational conditions of the transformed problem, and conversely.

5. Method of Lagrange multipliers. The preceding transformation of variational problems may be generalized when one employs the method of Lagrange multipliers. The minimum problem (1) with the restraints (2), (3) is replaced by an unrestrained variational problem: render stationary the expression

(19) $$I = \int_{x_o}^{x_1} [F + p(x) \left(\frac{du}{dx} - u'\right)] dx - q(u-\bar{u}) \Big|_{x_o}^{x_1} ,$$

where the mutually independent functions $u(x)$, $u'(x)$, $p(x)$ and the values q_o, q_1 are to be varied. The first variation of I is

$$(20) \qquad \delta I = \int_{x_o}^{x_1} [(F_u - \frac{dp}{dx})\delta u + (F_{u'} - p)\delta u' + (\frac{du}{dx} - u')\delta p] dx - [(q-p)\delta u + (u - \bar{u})\delta q \Big|_{x_o}^{x_1} .$$

Through variation of p and q the equations (2) $\frac{du}{dx} - u' = 0$, (3) $u - \bar{u} = 0$ at $x = x_o$, $x = x_1$ are obtained, and through variation of u and u',

$$(21) \qquad F_u - \frac{dp}{dx} = 0,$$

$$(22) \qquad F_{u'} - p = 0 ,$$

$$(23) \qquad q - p = 0, \;\; [\, q_o - p(x_o) = 0 \;,\;\; q_1 - p(x_1) = 0 \,].$$

The original problem arises when one initially imposes (2) and (3) as restraints. Then (21), (22) and (23) are variational conditions. On the other hand, if one places these last three conditions as restraints then (2) and (3) appear as variational conditions. This is the transformed maximum problem. Moreover, one makes use of the fact that this free variation of $p(x)$ is not restricted by the conditions (21), (22); however, this is no longer the case as soon as u and u' are expressed in terms of p and dp/dx in accord with (21), (22), as was supposed earlier.

As soon as $p(x)$ is introduced as a single new function, the substitution of (21), (22), (23) in (19), and integration by parts, yields the expression

$$\int_{x_o}^{x_1} [F - uF_u - u'F_{u'}] dx + F_{u'}\bar{u} \Big|_{x_o}^{x_1} \quad ,$$

which is the negative of (10).

Incidentally, it is to be noted that whenever only the equation (22) and the boundary conditions (3) are prescribed in problem (19) one is lead to the "canonical equations" (2), (21) of the variational problem.

To choose initially the Lagrange multiplier rule has the advantage that one sees readily how to proceed if further restraining conditions of an arbitrary nature are present.

Added in proof. As Mr. Courant has remarked, the passage from a minimum problem to a maximum problem has a purely conceptual basis. This will be discussed elsewhere.

III. Examples of variational problems in two variables.

1. We consider first the problem

$$(24) \qquad \frac{1}{2}\iint_G (u_x^2 + u_y^2 + au^2)\,dxdy = \text{Min.},\ u = \bar{u} \text{ on } \Gamma,\quad a > 0\,,$$

where G is a region in the (x, y)-plane with boundary Γ[9], and u(x, y) is a solution of this problem[10].

We introduce new functions p, q, d through the equations

$$(25) \qquad p = u_x\,,\quad q = u_y\,,\quad d = au,$$

and consider the new integrand

$$pu_x + qu_y + du - \frac{1}{2}(u_x^2 + u_y^2 + au^2) = \frac{1}{2}\left(\frac{1}{a}d^2 + p^2 + q^2\right).$$

Accordingly, we produce the new variational problem

$$(26) \qquad \frac{1}{2}\iint_G \left(\frac{1}{a}d^2 + p^2 + q^2\right)dxdy - \int_\Gamma (px_n + qy_n)\,\bar{u}\,ds = \text{Min.}$$

(x_n, y_n are components of the outer normal and s arc length on Γ), and corresponding to the Euler equation of (19), as a single subsidiary equation we impose

$$(27) \qquad d = p_x + q_y\,.$$

The Euler equations for (26) are then

$$(28) \qquad \begin{aligned} &\frac{1}{a}(p_x + q_y)_x - p = 0\,, \\ &\frac{1}{a}(p_x + q_y)_y - q = 0\,, \end{aligned}$$

and the natural boundary condition is

$$(29) \qquad \frac{1}{a}(p_x + q_y) - \bar{u} = 0 \quad \text{on } \Gamma.$$

These equations say that the function

$$u = \frac{1}{a}(p_x + q_y)$$

satisfies the following conditions:

$$u_x = p, \quad u_q = q, \quad \Delta u - au = 0, \quad u = \bar{u} \text{ on } \Gamma.$$

Again, the minimum value of (26) is equal to the negative of the minimum value of (24).

2. The Dirichlet problem

(30) $$\frac{1}{2}\iint_G (u_x^2 + u_y^2)\,dxdy = \text{Min.}, \quad u = \bar{u} \text{ on } \Gamma$$

corresponds to the degenerate case II. 3. Through the introduction of $u_x = p$, $u_y = q$, $\Phi(p,q) = \frac{1}{2}(p^2 + q^2)$ one attains the variational problem

(31) $$\frac{1}{2}\iint_G (p^2 + q^2)\,dxdy - \int_\Gamma (px_n + qy_n)\bar{u}\,ds = \text{Min.},$$

under the subsidiary equation $p_x + q_y = 0$. We satisfy this subsidiary equation by setting $p = v_y$, $q = -v_x$, and upon integrating by parts along Γ obtain the problem

(32) $$\frac{1}{2}\iint_G (v_x^2 + v_y^2)\,dxdy + \int_\Gamma v\bar{u}_s\,ds = \text{Min.},$$

whose solution v is the potential function conjugate to u. The equation $\Delta v = 0$ and the boundary condition $v_n = -\bar{u}_s$ are satisfied.

Without knowing the existence of a solution of (30), we can see that the sum of the greatest lower bounds of (30) and (32) is non-negative. This results from the identity

$$\iint_G (u_x^2 + u_y^2 + v_x^2 + v_y^2)\,dxdy + 2\int_\Gamma vu_s\,ds = \iint_G [(u_x - v_y)^2 + (u_y + v_x)^2]\,dxdy.$$

In the "free" problem (31) we can require that the functions p and q satisfy the associated Euler equation $p_y - q_x = 0$. We can then introduce a function u through the equations $u_x = p$, $u_y = q$, and obtain the problem

$$\frac{1}{2}\iint_G (u_x^2 + u_y^2)\,dxdy - \int_\Gamma u_n \bar{u}\,ds = \text{Min.} \tag{33}$$

for all functions satisfying the subsidiary equation $\Delta u = 0$. This is none other than the method considered by Trefftz. For numerical purposes it is perhaps practical to retain this supplementary procedure; however, whenever it is possible to obtain without difficulty approximation functions which as a matter of course satisfy the natural boundary conditions and not the Euler equation, one would be inclined to expect a better convergence when it is replaced by the Ritz method.

FOOTNOTES

1) [1), p. 13]. In general the proof of the convergence of a minimizing sequence provides some estimate of the approximation to the solution itself.

2) [2), p. 13]. Ein Gegenstück zum Ritzschen Verfahren Verhandlungen des zweiten Internationalen Kongresses für Technische Mechanik, Zürich 1927, S. 131, where also the numerical usefulness is examined in important practical cases. See in addition Konvergenz und Fehlerabschätzung beim Ritzschen Verfahren, Math. Annalen 101(1923), where also the pointwise approximation to the solution is deduced from the approximation to the minimum value.

3) [3), p. 13]. Indeed, this was the starting point of the following consideration. See Riemann, Weber and Frank, Die Differential- u. Integralgleichungen, II, V, § 4, 2. 3.

4) [Corrected version of a badly worded Footnote 1), p. 14]. The integrand function F of (1) is supposed to have continuous derivatives of the first two orders with respect to its arguments on a closed region B. The functions $u(x)$ and $u'(x)$, $x_0 \leq x \leq x_1$, are supposed to be bounded, with $(u'(x), u(x), x)$ in B. Also, conditions (4) and (5) hold in B.

5) [1), p. 15]. Instead of making the negative of (10) a maximum.

6) [Modified phrasing of Footnote 2), p. 15]. $(p'(x), p(x), x)$, $x_0 \leq x \leq x_1$ is supposed to lie in a region B^* of the same nature as B, and which is contained in the transform of B determined by (7); p' is supposed to be continuously differentiable.

7) [1), p. 16]. This corresponds to the Legendre transformation of the second variation.

8) [2), p. 16]. We write $F(u)$, $\Phi(p)$ in place of $F(u', u, x)$, $\Phi(p', p, x)$, etc.

9) [1), p. 19]. Perhaps with piecewise continuously differentiable tangent.

10) [2), p. 19]. It possesses continuous first and second derivatives on the interior of G.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Center for Cybernetic Studies The University of Texas	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Legacy of the Paper by Kurt Friedrichs Entitled Ein Verfahren Der Variationsrechnung, Das Minimum Eines Integral Als Das Maximum Eines Anderen Ausdruckes Darzustellen

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

William T. Reid

6. REPORT DATE August 1977	7a. TOTAL NO. OF PAGES 30	7b. NO. OF REFS 41

8a. CONTRACT OR GRANT NO. N00014-75-C-0569 b. PROJECT NO. NR047-021 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Center for Cybernetic Studies Research Report CCS 300 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Office of Naval Research (Code 434) Washington, D.C.

13. ABSTRACT

For certain aspects of the calculus of variations and optimal control theory, the paper of Kurt Friedrichs listed in the title of this report has had a strong influence, both directly and indirectly. The paper is devoted to a brief historical survey of this influence, especially in the area of convex analysis. In addition, a translation of Friedrichs' paper is appended to this report.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

Unclassified

Security Classification

A-31408

Unclassified

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Calculus of variations						
reciprocal problems						
Problems of Bolza						
Duality						
Convexity						

DD FORM 1 NOV 65 1473 (BACK)

S/N 0102-014-6800

Unclassified

Security Classification